Bayesian Inverse Problem for space–time DSD

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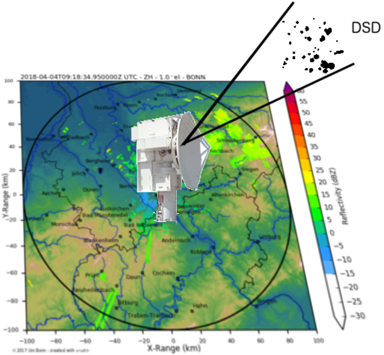
Terrestrial Systems Research: Monitoring, Prediction and High Performance Computing

RADAR OBSERVATIONS

- (polarimetric) radar observations
- observed quantity: drop size distribution (DSD)
- polar coordinates: radar bins
- space-time unknowns
- finitely many nonlinear observations



Observation operator = polarimetric radar

$$= \left(\int_{D_{\min}}^{D_{\max}} f_1(D) N(D) dD, \ldots, \int_{D_{\min}}^{D_{\max}} f_4(D) N(D) dD \right)^\top$$

where

Drop size distribution (DSD)

$$N(D; x, t) = \begin{cases} N_0(x,t) D^{\mu(x,t)} \exp\left(-\Lambda(x,t) D\right), & \gamma - \text{distribution} \\ N_0(x,t) \exp\left(-\Lambda(x,t) D\right), & \mu \equiv 0, \text{ Marshall-Palmer} \end{cases}$$

- $N(D)$ drop size distribution (DSD) $[m^4]$
- $D_{\min} \leq D \leq D_{\max}$ drop diameter $[m]$
- N_0 intercept parameter $[m^{-(M+4)}]$
- Λ slope $[m^{-1}]$
- M dimensionless shape parameter

Polarimetric forward operator I

- horizontal reflectivity

$$Z_h = \frac{4\lambda^4}{\pi^4 |K_w|^2} \int_{D_{min}}^{D_{max}} |f_{HH}(\pi, D)|^2 N(\Lambda, N_0, \mu, D) dD \; \left[\mathrm{mm}^6\mathrm{m}^{-3}\right]$$

- vertical reflectivity by

$$Z_v = \frac{4\lambda^4}{\pi^4 |K_w|^2} \int_{D_{min}}^{D_{max}} |f_{VV}(\pi, D)|^2 N(\Lambda, N_0, \mu, D) dD \; \left[\mathrm{mm}^6\mathrm{m}^{-3}\right]$$

where

- λ is the radar wavelength in $[\mathrm{cm}]$
- K_w is the dielectric constant of water

POLARIMETRIC FORWARD OPERATOR II

- specific differential phase

$$K_{DP} = \frac{180\lambda}{\pi} \int_{D_{min}}^{D_{max}} \mathrm{Re}(f_{HH}(0,D) - f_{VV}(0,D))N(D)dD \left[\mathrm{deg\ km}^{-1}\right].$$

- cross-correlation between vertical and horizontal reflectivity

$$\rho_{HV} = \frac{\pi^4 |K_w|^2}{4\lambda^4} \frac{1}{\sqrt{Z_h Z_v}} \int_{D_{min}}^{D_{max}} f_{HH}^*(\pi,D) f_{VV}(\pi,D)N(D)dD$$

where

- $f_{XX}(\pi, D)(f_{XX}(0, D))$ the back (forward)-scattering amplitudes
- indices HH (or VV) the horizontal (vertical) polarization for both receiving and transmitting

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Comparison of Polarimetric Radar Drop Size Distribution Retrieval Algorithms, E.A. Brandes, G. Zhang, and J. Vivekanandan, J. Atmos. Oceanic Technol., 2004

General setting

- $\mathbf{X}$ real separable Hilbert space (Banach also possible)
- $\mathbf{Y} \cong \mathbb{R}^N$ finite dimensional vector space
- $\eta \sim \mathcal{N}(0, E)$ observational noise
- $G : \mathbf{X} \to \mathbf{Y}$ observation operator

$$y = G(x) + \eta$$

- Goal: get information on x from observation $y \in \mathbb{R}^N$.
- Question 1: What kind of information? conditional measures
- Question 2: Well-posed? No. Prior regularizes.

General Bayes

- potential

$$\Phi(x;y) := \frac{1}{2}\left\|E^{-\frac{1}{2}}\left(y - G(x)\right)\right\|$$

- posterior probability measure $\pi^{(y)}$, for x given y:

$$\frac{d\pi^{(y)}}{d\pi^{(0)}} = \frac{1}{Z(y)}\exp\left(-\Phi(x;y)\right)$$

- normalization

$$Z(y) := \int_{\mathbf{X}} \exp\left(-\Phi(x;y)\right) d\pi^{(0)}(x)$$

Inverse problems: A Bayesian perspective, A. M. Stuart, Acta Numerica, 2010

PRIOR

GAUSSIAN MEASURES ON SEPARABLE HILBERT SPACES

- $\mathcal{H}$ separable Hilbert space
- $\pi_0 \sim \mathcal{N}(m_0, \mathcal{C}_0)$, centered if $m_0 = 0$ (from now on)
- $\mathcal{C}_0$ self-adjoint, positive semi-definite trace-class operator on $\mathcal{H}$
- Cameron-Martin $\mathcal{E} = \text{Range}(\mathcal{C}_0)^{\frac{1}{2}})$
- $\mathcal{E} \subset \mathcal{X} \subset \mathcal{H}$ with $\nu(\mathcal{X}) = 1$

- space time domain $\Omega = [0, T] \times [0, H] \times D$, $D \subset \mathbb{R}^d$ compact
- $\mathcal{H} = L_2(\Omega)$
- $\mathcal{X} = H^{s_1, s_2, s_3}(\Omega)$ tensor product Sobolev space, $s_i > d_i/2$
- $\mathcal{C}_0 f = \int_\omega K(x, y) f(y) dy$ $K : \Omega \times \Omega$ reproducing kernel
- $\mathcal{E} = H^{\sigma_1, \sigma_2, \sigma_3}(\Omega)$, with $\sigma_i > s_i$

Well-posed observation operator

$$G : \mathcal{X} \to \mathbb{R}^{4 \times m_T \times M}, \quad G_i(N) = \int_{D_{\min}}^{D_{\max}} f_i(D) N(D)(t_\ell, x_k) dD$$

- $\forall \epsilon > 0 \exists M(\epsilon)$ s.t. $\forall x \in \mathbf{X}$

$$\left\| E^{-\frac{1}{2}} G(x) \right\| \leq \exp(\epsilon \|x\|_{\mathbf{X}}^2 + M)$$

- $\forall r > 0 \exists K(r)$ s. t. $\forall x_1, x_2 \in \mathbf{X}$ with $\max_{i=1,2}\{\|x_i\|\} \leq r$, we have

$$\left\| E^{-\frac{1}{2}} \left(G(x_1) - G(x_2) \right) \right\| \leq K(r) \|x_1 - x_2\|_{\mathbf{X}}^2$$

The polarimetric radar observation operator is well-posed.

Inverse problems: A Bayesian perspective, A. M. Stuart, Acta Numerica, 2010

Approximation of posterior

- approximate potential

$$\Phi_\Xi(x;y) := \frac{1}{2}\left\| E^{-\frac{1}{2}}\left(y - G\left(\sum_{\xi\in\Xi}\alpha_\xi K(\cdot,\xi)\right)\right)\right\|$$

- need $\psi(\Xi) \to 0$ as $\Xi \subset \Omega$ gets dense s.t.

$$\left| G(x) - G\left(\sum_{\xi\in\Xi}\alpha_\xi K(\cdot,\xi)\right)\right| \leq K(\epsilon)\exp(\epsilon\|x\|^2_{\mathcal{X}})\psi(\Xi)$$

- error in Hellinger distance $d_{\text{Hel}}(\pi, \pi_\Xi) \leq C\psi(\Xi)$
- (approximate) MAP-estimator

$$u_{\text{MAP}} = \arg\min_{u\in\mathcal{E}} \frac{1}{2}\|u\|^2_{\mathcal{E}} + \Phi(u;y)$$

Linear observations

Generalized reconstruction

- Given unknown function $u \in \mathcal{H}(D)$, data $y_j := \lambda_j(u) \in \mathbb{R}$, $1 \leq j \leq N$
- Aim function $s \in \mathcal{H}(D)$, s.t. $\lambda_j(s) \approx \lambda_j(u)$ and $\|s\|_{\mathcal{H}(D)}$ bounded

- $\Lambda_N := \{\lambda_1, \ldots, \lambda_N\} \subset \mathcal{H}'(D)$ linearly independent.
- Riesz representer:

$$\mathcal{R}_\lambda(x) = (\mathcal{R}_\lambda, K(\cdot, x))_{\mathcal{H}(D)} = \lambda^{(1)} K(\cdot, x)$$

- Regression functional

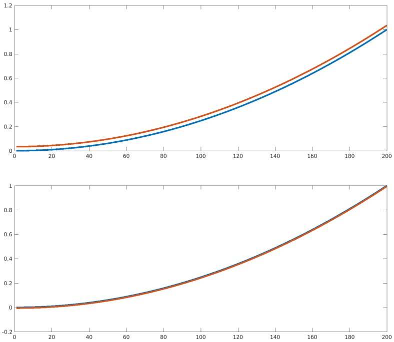
$$J_{\alpha;\Lambda_N}(s) := \sum_{j=1}^{N} \left(\lambda_j(s) - \lambda_j(u)\right)^2 + \alpha \|s\|^2_{\mathcal{H}(D)}$$

- deterministic error analysis available

Wendland + CR, 2005, Zwicknagl+CR 2014, Griebel +CR 2017, Kempf Wendland + CR 2018

NUMERICAL (TOY) EXAMPLE

- observation operator
  $\int_{0.1}^{2} z^2 \exp(\rho + \mu \ln(z) - \lambda z)dz$
- $\Xi_N \subset [0, 1]$, $N = 20, 60$
- approximation
  $\rho, \mu\lambda \approx \sum_{\xi} \alpha_{\xi} K(\cdot, \xi)$
  Whittle-Matérn, smoothness 2
- white noise, small variance
- right: reconstruction μ with $N = 20$ upper



Conclusion & Outlook

- infinite dimensional Bayesian inversion
- well-posed problem
- error analysis
- MAP estimator

To Do:

- radar forward operator, T-matrix (K. Schinagl)
- MCMC with suitable proposal densities